CHARACTERIZING PROJECTIVE SPACES ON DEFORMATIONS OF HILBERT SCHEMES OF K3 SURFACES

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1. INTRODUCTION

Let X be an irreducible holomorphic symplectic manifold, i.e., a compact Kähler simply-connected manifold admitting a unique nondegenerate holomorphic two-form. Let $(,)$ denote the Beauville–Bogomolov form on the cohomology group $\mathrm{H}^2(X, \mathbb{Z})$, normalized so that it is integral and primitive. When X is a K3 surface this coincides with the intersection form. In higher dimensions, the form induces an inclusion

$$\text{(1)} \qquad \mathrm{H}^2(X, \mathbb{Z}) \subset \mathrm{H}_2(X, \mathbb{Z}),$$

which allows us to extend $(,)$ to a $\mathbb{Q}$-valued quadratic form.

Lagrangian projective spaces play a fundamental rôle in the birational geometry of these classes of manifolds. If X contains a holomorphically embedded projective space $\mathbb{P}^{\dim(X)/2}$ we can consider the *Mukai flop* of X, obtained by blowing up the projective space and blowing down the exceptional divisor

$$E \simeq \mathbb{P}(\Omega^1_{\mathbb{P}^{\dim(X)/2}})$$

along the opposite ruling. Our goal is to characterize possible homology classes of such submanifolds, modulo the monodromy representation on the cohomology of X.

Assuming X contains a Lagrangian projective space $\mathbb{P}^{\dim(X)/2}$, let $\ell \in \mathrm{H}_2(X, \mathbb{Z})$ denote the class of a line in $\mathbb{P}^{\dim(X)/2}$, and $\lambda = N\ell \in \mathrm{H}^2(X, \mathbb{Z})$ a positive integer multiple. We can take N to be the index of $\mathrm{H}^2(X, \mathbb{Z}) \subset \mathrm{H}_2(X, \mathbb{Z})$. Hodge theory [17, 22] shows that the deformations of X containing a deformation of the Lagrangian space coincide with the deformations of X for which $\lambda \in \mathrm{H}^2(X, \mathbb{Z})$ remains of type $(1, 1)$. Infinitesimal Torelli implies this is a divisor in the deformation space, i.e.,

$$\lambda^\perp \subset \mathrm{H}^1(X, \Omega^1_X) \simeq \mathrm{H}^1(X, \mathcal{T}_X).$$

We seek to establish intersection theoretic properties of ℓ for various deformation-equivalence classes of holomorphic symplectic manifolds. Previous results in this direction include

(1) If X is a K3 surface then $(\ell, \ell) = -2$.
(2) If X is deformation equivalent to the Hilbert scheme of length-two subschemes of a K3 surface then $(\ell, \ell) = -5/2$. [11]
(3) If X is deformation equivalent to a generalized Kummer fourfold then $(\ell, \ell) = -3/2$. [12]

Here we prove

Theorem 1.1. *Let X be a six-dimensional Kähler manifold, deformation equivalent to the Hilbert scheme of length-three subschemes of a K3 surface. Let $\mathbb{P}^3 \subset X$ be a smooth subvariety and $\ell \subset \mathbb{P}^3$ a line. Then $(\ell, \ell) = -3$ and $\rho = 2\ell \in \mathrm{H}^2(X, \mathbb{Z})$. Furthermore, we have*

$$\left[\mathbb{P}^3 \right] = \frac{1}{48} \left(\rho^3 + \rho^2 c_2(X) \right).$$

This uniquely characterizes the class of the Lagrangian plane, modulo the monodromy action, which acts transitively on the $\rho \in \mathrm{H}^2(X, \mathbb{Z})$ with $(\rho, \rho) = -12$ and $(\rho, \mathrm{H}^2(X, \mathbb{Z})) = 2\mathbb{Z}$ [8, §3].

In general, we conjectured in [10] that if X is of dimension $2n$ then $(\ell, \ell) = -(n+3)/2$, if X is deformation equivalent to a Hilbert scheme of a K3 surface. Our main motivation for making these conjectures is to achieve a classification of extremal rational curves on irreducible holomorphic symplectic varieties (i.e., generators of extremal rays of birational contractions) in terms of intersection properties under the Beauville-Bogomolov form.

The structure of this paper is as follows: Section 2 reviews the cohomology groups of Hilbert schemes of K3 surfaces; Section 3 focuses on the ring structure. We employ representation theory to get results on the Hodge classes in Section 4. The Hilbert scheme of length-three subschemes is studied in detail in Section 5. We extract the distinguished absolute Hodge class in the middle cohomology in Section 6; here 'absolute Hodge classes' are those that remain Hodge under arbitrary deformations of complex structure. The computation of the class of the Lagrangian three planes is worked out in Section 7, modulo a number theoretic result. This is proved in Section 8.

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2. Cohomology of Hilbert schemes

Let X be deformation equivalent to the punctual Hilbert scheme $S^{[n]}$, where S is a K3 surface. For $n > 1$ the Beauville-Bogomolov form can be written [1, §8]

$$\mathrm{H}^2(X, \mathbb{Z}) \simeq \mathrm{H}^2(S, \mathbb{Z})_{(,)} \oplus_{\perp} \mathbb{Z}\delta, \quad (\delta, \delta) = -2(n - 1)$$

where 2δ is the class of the 'diagonal' divisor $\Delta^{[n]} \subset S^{[n]}$ parameterizing nonreduced subschemes. For each homology class $f \in \mathrm{H}^2(S, \mathbb{Z})$, let $f \in \mathrm{H}^2(X, \mathbb{Z})$ denote the class parameterizing subschemes with some support along f. This is compatible with the lattice embedding above. Duality gives a $\mathbb{Q}$-valued form on homology

$$\mathrm{H}_2(X, \mathbb{Z}) \simeq \mathrm{H}_2(S, \mathbb{Z})_{(,)} \oplus_{\perp} \mathbb{Z}\delta^{\vee}, \quad (\delta^{\vee}, \delta^{\vee}) = -\frac{1}{2(n - 1)},$$

where $\delta^{\vee}$ is characterized as the homology class orthogonal to $\mathrm{H}^2(S, \mathbb{Z})$ and satisfying $\delta^{\vee} \cdot \delta = 1$.

Theorem 2.1. [7] *Let S be a K3 surface and $S^{[n]}$ its Hilbert scheme. Consider the Poincaré polynomial*

$$p(S^{[n]}, z) = \sum_{j=0}^{4n} \beta_j(S^{[n]}) z^j.$$

Then

$$\sum_{n=0}^{\infty} p(S^{[n]}, z) t^n = \prod_{m=1}^{\infty} (1 - z^{2m-2} t^m)^{-1} (1 - z^{2m} t^m)^{-22} (1 - z^{2m+2} t^m)^{-1}.$$

To save space, we write

$$q(S^{[n]}, z) = \sum_{j=0}^{n} \beta_{2j} z^j,$$

which determines the Poincaré polynomial by Poincaré duality. We have

$$
\begin{array}{rcl}
q(S, z) & = & 1 + 22z \\
q(S^{[2]}, z) & = & 1 + 23z + 276z^2 \\
q(S^{[3]}, z) & = & 1 + 23z + 299z^2 + 2554z^3.
\end{array}
$$

A theorem of Verbitsky [21, Theorem 1.5] asserts that the homomorphism arising from the cup product

$$\mu_{k,n} : \mathrm{Sym}^k \mathrm{H}^2(S^{[n]}, \mathbb{Q}) \to \mathrm{H}^{2k}(S^{[n]}, \mathbb{Q})$$

is injective for $k \leq n$. Thus its image has dimension
$$\binom{22 + k}{k}.$$
In light of the computations above, $\mu_{2,2}$ is an isomorphism, $\mu_{2,3}$ has cokernel of dimension 23, and $\mu_{3,3}$ has cokernel of dimension
$$2554 - \binom{25}{3} = 254 = \binom{23}{2} + 1.$$
The cup product also induces a homomorphism
$$\mathrm{coker}(\mu_{2,3}) \otimes \mathrm{H}^2(S^{[3]}, \mathbb{Q}) \to \mathrm{coker}(\mu_{3,3}).$$
This homomorphism has been observed by Markman [15, p. 80]. More generally, he analyzes what classes are needed to generate the cohomology ring $\mathrm{H}^*(S^{[n]}, \mathbb{Q})$, beyond those coming $\mathrm{H}^2(S^{[2]}, \mathbb{Q})$. Markman uses Chern classes of universal sheaves over the product $S^{[n]} \times S$; a detailed discussion of the $n = 3$ case is given in [15, Ex. 14].

3. THE RING STRUCTURE ON COHOMOLOGY

Lehn-Sorger [13] and Nakajima [16] described $\mathrm{H}^*(S^{[n]}, \mathbb{Q})$ in terms of $\mathrm{H}^*(S, \mathbb{Q})$. We review the Lehn-Sorger formalism for the cup product on the cohomology ring.

Let S be a K3 surface and $A = \mathrm{H}^*(S, \mathbb{Q})(1)$, the cohomology ring shifted so that it has weights $-2, 0$, and 2; this is written as $\mathrm{H}^*(S, \mathbb{Q})[2]$ in their paper. Shifting the weights changes the sign of the intersection form, which is denoted by $\langle , \rangle$; this has signature $(20, 4)$. Let $T : A \to \mathbb{Q}$ denote the linear form
$$\gamma \mapsto - \int_S \gamma$$
and $\langle , \rangle$ the induced bilinear form
$$\langle \gamma_1, \gamma_2 \rangle = T(\gamma_1 \gamma_2) = - \int_S \gamma_1 \gamma_2.$$

For each $n \in \mathbb{N}$, we endow $A^{\otimes n}$ with an analogous structure. We shall use the fact that A has only graded pieces of *even* degrees to simplify the description in [13]. In this situation, graded commutative multiplication rules are in fact commutative, given by the rule
$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_n) = (a_1 b_1) \otimes \cdots \otimes (a_n b_n).$$
The linear form
$$T : A^{\otimes n} \to \mathbb{Q}$$
is defined by
$$T(a_1 \otimes \cdots \otimes a_n) = T(a_1) \cdots T(a_n).$$

Let $\langle , \rangle$ denote the associated bilinear form

$$\langle a, b \rangle = T(a \cdot b).$$

The symmetric group $\mathfrak{S}_n$ acts on $A^{\otimes n}$ by the rule

$$\pi(a_1 \otimes \cdots \otimes a_n) = a_{\pi^{-1}(1)} \otimes \cdots \otimes a_{\pi^{-1}(n)}.$$

Given a partition $n = n_1 + \ldots + n_k$ with $n_1, \ldots, n_k \in \mathbb{N}$, we have a generalized multiplication map

$$
\begin{array}{rcl}
A^{\otimes n} & \to & A^{\otimes k} \\
a_1 \otimes \cdots \otimes a_n & \mapsto & (a_1 \cdots a_{n_1}) \otimes \cdots \otimes (a_{n_1 + \cdots + n_{k-1} + 1} \cdots a_{n_1 + \cdots + n_k}).
\end{array}
$$

Given a finite set $I \subset \{1, \ldots, n\}$, let $A^{\otimes I}$ denote the tensor power with factors indexed by elements of I. Given a surjection $\phi : I \to J$, there is an induced multiplication

$$\phi^* \colon A^{\otimes I} \to A^{\otimes J}$$

defined as above. Let

$$\phi_* : A^{\otimes J} \to A^{\otimes I}$$

denote the *adjoint* of ϕ^*, i.e.,

$$\langle \phi^* a, b \rangle = \langle a, \phi_* b \rangle$$

for $a \in A^{\otimes I}$ and $b \in A^{\otimes J}$.

We have the composite

$$A \xrightarrow{\Delta_*} A \otimes A \to A,$$

where the first map is adjoint comultiplication and the second is multiplication. Let $e := e(A)$ denote the image of 1 under the composed map.

Remark 3.1. We evaluate the signs of $\Delta_* 1$ and $e(A)$. Let Δ_S denote the fundamental class of the diagonal in $\mathrm{H}^*(S \times S, \mathbb{Z}) = \mathrm{H}^*(S, \mathbb{Z}) \otimes \mathrm{H}^*(S, \mathbb{Z})$. Using the adjoint property, we have

$$
\begin{array}{rcl}
\langle \Delta_* 1, \alpha \otimes \beta \rangle & = & \langle 1, \alpha\beta \rangle \\
& = & T(\alpha\beta) \\
& = & -\int_S \alpha\beta
\end{array}
$$

whereas

$$
\begin{array}{rcl}
\langle \Delta_S, \alpha \otimes \beta \rangle & = & \left\langle \sum_j e_j \otimes e_j^\vee, \alpha \otimes \beta \right\rangle \\
& = & \sum_j T(e_j \alpha) T(e_j^\vee \beta) \\
& = & \int_S \alpha\beta,
\end{array}
$$

where $\{e_j\}$ is a homogeneous basis for $\mathrm{H}^*(S, \mathbb{Q})$ with Poincaré-dual basis $e_j^\vee$. Therefore, we find

$$(2) \qquad\qquad \Delta_* 1 = -[\Delta_S].$$

Furthermore, we have

$$\int_S e(A) = -T(e(A)) = -\langle e(A), 1 \rangle = -\langle \Delta_* 1, \Delta_* 1 \rangle = -\chi(S) = -24,$$

so $e(A)$ is a *negative* multiple of the point class. Nevertheless, we still have (cf. [13, §2.2])

$$e(A) = \chi(S)\text{vol}, \quad \text{where} \quad T(\text{vol}) = 1,$$

but vol differs from the standard volume form by sign.

Let $\langle \pi \rangle \backslash [n]$ denote the set of orbits of $[n] = \{1, 2, \ldots, n\}$ under the action of π. Set

$$A\{\mathfrak{S}_n\} = \oplus_{\pi \in \mathfrak{S}_n} A^{\otimes \langle \pi \rangle \backslash [n]} \cdot \pi$$

which admits an action of $\mathfrak{S}_n$. First, note that $\sigma \in \mathfrak{S}_n$ induces a bijection

$$\begin{array}{rcl} \sigma : \langle \pi \rangle \backslash [n] & \to & \langle \sigma \pi \sigma^{-1} \rangle \backslash [n] \\ x & \mapsto & \sigma x. \end{array}$$

Thus we obtain an isomorphism

$$\begin{array}{rcl} \tilde{\sigma} : A\{S_n\} & \to & A\{S_n\} \\ a\pi & \mapsto & \sigma^* \sigma \pi \sigma^{-1}. \end{array}$$

Example 3.2. [13, 2.9, 2.17] We have $A\{\mathfrak{S}_2\} = A^{\otimes 2}\text{id} \oplus A(12)$ and

$$A\{\mathfrak{S}_3\} = A^{\otimes 3}\text{id} \oplus A^{\otimes 2}(12) \oplus A^{\otimes 2}(13) \oplus A^{\otimes 2}(23) \oplus A(123) \oplus A(132).$$

Let $A^{[n]} \subset A\{\mathfrak{S}_n\}$ denote the invariants under this action. Then we have [13, §2]

$$A^{[n]} = \sum_{\|\alpha\|=n} \bigotimes_i \text{Sym}^{\alpha_i} A,$$

where α corresponds to a partition

$$\underbrace{1 + \cdots + 1}_{\alpha_1 \text{ times}} + \underbrace{2 + \cdots + 2}_{\alpha_2 \text{ times}} + \cdots$$

and

$$n = \|\alpha\| = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n.$$

Note that this is compatible with Hodge structures; in particular, $A^{[n]}$ is a representation of the Hodge group of S and the special orthogonal group G_S associated with the intersection form on $\text{H}^2(S, \mathbb{R})$. We interpret this as acting on A, trivially on the summands $\text{H}^0(S, \mathbb{R})$ and $\text{H}^4(S, \mathbb{R})$.

Theorem 3.3. [13, Theorem 3.2] *Let S be a K3 surface. Then there is a canonical isomorphism of graded rings*

$$(\text{H}^*(S, \mathbb{Q})[2])^{[n]} \overset{\sim}{\to} \text{H}^*(S^{[n]}, \mathbb{Q})[2n].$$

In the cohomology of the Hilbert scheme, the subring generated by $\mathrm{H}^2(S^{[n]})$ plays a special role. We have an isomorphism

$$\mathrm{H}^2(S^{[n]}, \mathbb{Z}) = \mathrm{H}^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

where 2δ parameterizes the non-reduced schemes of S. We express this in terms of our presentation. Given $D \in \mathrm{H}^2(S, \mathbb{Z})$, the class

$$\sum_{i=1}^{n} 1_{\{1\}} \otimes \cdots \otimes 1_{\{i-1\}} \otimes D_{\{i\}} \otimes 1_{\{i+1\}} \otimes \cdots \otimes 1_{\{n\}}(\mathrm{id})$$

is the corresponding class in $\mathrm{H}^2(S^{[n]}, \mathbb{Q})[2n]$. Using the explicit form of the isomorphism in [13, 2.7] and Nakajima's isomorphism ([13, Thm. 3.6]), we find that

$$\delta = \sum_{1 \leq i < j \leq n} 1_{\{1\}} \otimes \ldots \otimes 1_{\{i-1\}} \otimes 1_{\{i,j\}} \otimes 1_{\{i+1\}} \otimes \cdots \otimes 1_{\{j-1\}} \otimes 1_{\{j+1\}} \otimes \cdots \otimes 1_{\{n\}}(ij).$$

Here is the essence of the computation: the interpretation of the nonreduced subschemes via the correspondence

$$Z_2 = \{(\xi, x, \xi') : |\xi'| - |\xi| = 2x\} \subset S^{[n-2]} \times S \times S^{[n]}$$

allows us to express δ in terms of Nakajima's creation and annihilation operators, and thus in

$$\mathrm{H}^*(S^{[n]}, \mathbb{Q})[2n].$$

We describe the general rule for evaluating the fundamental class in $A^{[n]}$. Let

$$[\mathrm{pt}] \in \mathrm{H}^4(S, \mathbb{Z})[2] \subset A$$

be the point class, which is of degree -2. Let

$$[\mathrm{pt}]_{\{1\}} \otimes \cdots \otimes [\mathrm{pt}]_{\{n\}}(\mathrm{id}) \in A^{[n]}$$

denote the unique class of degree $-2n$ up to scalar. Then the class of a point in $S^{[n]}$ is equal to [13, 2.10]

$$(3) \qquad [\mathrm{pt}_{S^{[n]}}] = \frac{1}{n!} [\mathrm{pt}]_{\{1\}} \otimes \cdots \otimes [\mathrm{pt}]_{\{n\}}(\mathrm{id}).$$

4. Decomposition of the cohomology representation

We summarize general results from representation theory. For an orthogonal group of odd dimension $2r + 1$, the highest weights $\lambda = (\lambda_1, \ldots, \lambda_r)$ of irreducible representations $V(\lambda)$ are vectors consisting entirely of integers (or half integers) in the fundamental chamber

$$\{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{r-1} \geq \lambda_r \geq 0\}.$$

Since we only consider even-weight representations, we ignore cases where the λ_j are half-integers. For orthogonal groups of even dimension $2r$, the fundamental chamber is

$$\{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0\}.$$

Recall that

- $V(1, 0, \ldots)$ is the standard representation V.
- We have

$$V(\underbrace{1, \cdots, 1}_{k \text{ times}}, 0, \cdots) = \bigwedge^k V,$$

  provided $k < r$ (in the even case) or $k \leq r$ (in the odd case); see, for instance, [6, Thms. 19.2 and 19.14].
- $V(k, 0, \ldots) = \mathrm{Sym}^k(V)/\mathrm{Sym}^{k-2}(V)$, embedded via the dual to the quadratic form on V.
- For the odd orthogonal group, we have

$$\dim V(\lambda) = \prod_{i<j} \frac{\ell_i - \ell_j}{j - i} \prod_{i \leq j} \frac{\ell_i + \ell_j}{2n + 1 - i - j}$$

  where $\ell_i = \lambda_i + n - i + \frac{1}{2}$ [6, p. 408].
- For the even orthogonal group, we have

$$\dim V(\lambda) = \prod_{i<j} \frac{\ell_i^2 - \ell_j^2}{(j-i)(2n-i-j)}$$

  where $\ell_i = \lambda_i + n - i$ [6, p. 410].
- Let $V_X(\lambda)$ denote an irreducible representation of an orthogonal group G_X of dimension $2r + 1$, $G_S \subset G_X$ the orthogonal subgroup $G_S \subset G_X$ of dimension $2r$ fixing a non-isotropic vector with negative self-intersection, and $V_S(\overline{\lambda})$ the representation of G_S with highest weight $\overline{\lambda}$. Then we have the branching rule [6, p. 426]

$$\mathrm{Res}^{G_X}_{G_S} V_X(\lambda) = \oplus_{\overline{\lambda}} V_S(\overline{\lambda}),$$

  where the sum ranges over all $\overline{\lambda}$ with

$$\lambda_1 \geq \overline{\lambda}_1 \geq \lambda_2 \geq \overline{\lambda}_2 \geq \cdots \lambda_r \geq |\overline{\lambda}_r|.$$

Let X be a generic deformation of $S^{[n]}$. Our goal is to decompose $\mathrm{H}^*(X, \mathbb{Q})$ into irreducible representations for the action of the identity component G_X of the special orthogonal group associated with the Beauville-Bogomolov form on $\mathrm{H}^2(X, \mathbb{Q})$. Let G_S denote the identity

component of the special orthogonal group associated with the intersection form on $H^2(S, \mathbb{Q})$. The decomposition

$$H^2(S^{[2]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta$$

induces an inclusion $G_S \subset G_X$.

Proposition 4.1. *Let X be deformation equivalent to $S^{[n]}$ for some n. Then G_X admits a representation on the cohomology ring of X.*

Proof. Let $\mathrm{Mon} \subset \mathrm{Aut}(H^*(X, \mathbb{Z}))$ denote the monodromy group, i.e., the group generated by the monodromy representations of all connected families containing X. Let $\mathrm{Mon}^2 \subset \mathrm{Aut}(H^2(X, \mathbb{Z}))$ denote its image under projection to the second cohomology group, so we have an exact sequence

$$1 \to K \to \mathrm{Mon} \to \mathrm{Mon}^2 \to 1.$$

Markman has shown [14, §4.3] that K is finite.

Note that G_X is a connected component of the Zariski closure of Mon^2 (see, for example [14, §1.8]). Since Mon and Mon^2 differ only by finite subgroups, it follows that the universal cover $\widetilde{G_X} \to G_X$ acts on the cohomology ring of X. Since the cohomology of X is nonzero only in even degrees, this representation passes to G_X. $\square$

In principle, we can decompose $H^*(X, \mathbb{R})$ explicitly into isotypic components as follows:

(1) Fix an embedding $G_S \subset G_X$, e.g., using the isomorphism

$$H^2(X, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta,$$

and compatible maximal tori (both of which have rank 11).
(2) Identify the highest-weight irreducible G_S-representation $V_S(\lambda) \subset H^*(S^{[n]}, \mathbb{R})$, which is a summand of the restriction of an irreducible $V_X(\lambda) \subset H^*(X, \mathbb{R})$. Decompose $V_X(\lambda)$ into irreducible G_S-representations.
(3) Repeat step two for $H^*(X, \mathbb{R})/V_X(\lambda)$ and subsequent quotients.

First consider $X = S^{[2]}$. We have decompositions

$$H^*(S^{[2]}) = A \oplus \mathrm{Sym}^2(A)$$

inducing

$$
\begin{aligned}
H^2(S^{[2]}) &= H^0(S) \oplus (H^0(S) \otimes H^2(S)) = \mathbf{1}_S \oplus V_S(1, 0, \ldots) \\
H^4(S^{[2]}) &= H^2(S) \oplus (H^0(S) \otimes H^4(S)) \oplus \mathrm{Sym}^2(H^2(S)) \\
&= V_S(1, 0, \ldots) \oplus \mathbf{1}_S^{\oplus 2} \oplus V_S(2, 0, \ldots)
\end{aligned}
$$

Let $V_X(2, 0, \ldots, 0)$ denote the highest-weight representation associated to $\mathrm{Sym}^2(\mathrm{H}^2(X))$ so that

$$\mathrm{Sym}^2(\mathrm{H}^2(X)) = V_X(2, 0, \ldots) \oplus \mathbf{1}_X.$$

The branching rule gives

$$V_X(1, 0, \ldots) = V_S(1, 0, \ldots) \oplus \mathbf{1}_S$$

and

$$V_X(2, 0, \ldots) = V_S(2, 0, \ldots) \oplus V_S(1, 0, \ldots) \oplus \mathbf{1}_S.$$

Therefore we obtain

$$\begin{array}{rcl}
\mathrm{H}^2(X) & = & V_X(1, 0, \ldots) \\
\mathrm{H}^4(X) & = & V_X(2, 0, \ldots) \oplus \mathbf{1}_X.
\end{array}$$

Now consider $X = S^{[3]}$. We have

$$\mathrm{H}^*(S^{[3]}) = A \oplus (A \otimes A) \oplus \mathrm{Sym}^3(A)$$

inducing following decompositions (as described in [13, Example 2.9]):

$$\begin{array}{rcl}
\mathrm{H}^2(S^{[3]}) & = & (\mathrm{H}^0(S)^{\otimes 2}) \oplus (\mathrm{H}^2(S) \otimes \mathrm{H}^0(S)^{\otimes 2}) \\
& = & \mathbf{1}_S \oplus V_S(1, 0 \ldots) \\
\mathrm{H}^4(S^{[3]}) & = & \mathrm{H}^0(S) \oplus (\mathrm{H}^0(S) \otimes \mathrm{H}^2(S))^{\oplus 2} \\
& & \oplus (\mathrm{Sym}^2(\mathrm{H}^2(S)) \otimes \mathrm{H}^0(S)) \oplus (\mathrm{H}^4(S) \otimes \mathrm{H}^0(S)^{\otimes 2}) \\
& = & \mathbf{1}_S^{\oplus 3} \oplus V_S(1, 0, \ldots)^{\oplus 2} \oplus V_S(2, 0, \ldots) \\
\mathrm{H}^6(S^{[3]}) & = & \mathrm{H}^2(S) \oplus (\mathrm{H}^2(S) \otimes \mathrm{H}^2(S)) \oplus (\mathrm{H}^0(S) \otimes \mathrm{H}^4(S))^{\oplus 2} \\
& & \oplus \mathrm{Sym}^3(\mathrm{H}^2(S)) \oplus (\mathrm{H}^4(S) \otimes \mathrm{H}^2(S) \otimes \mathrm{H}^0(S)) \\
& = & \mathbf{1}_S^{\oplus 3} \oplus V_S(1, 0, \ldots)^{\oplus 3} \oplus V_S(1, 1, 0, \ldots) \\
& & \oplus V_S(2, 0, \ldots) \oplus V_S(3, 0, \ldots).
\end{array}$$

Let $V_X(1, 1, 0, \ldots) = \bigwedge^2 V_X(1, 0, \ldots)$ and $V_X(3, 0, \ldots)$ denote the highest weight representation in $\mathrm{Sym}^3(V_X(1, 0, \ldots))$ so that

$$\mathrm{Sym}^3(V_X(1, 0, \ldots)) = V_X(3, 0, \ldots) \oplus V_X(1, 0, \ldots).$$

Therefore we obtain

$$\begin{array}{rcl}
\mathrm{H}^2(X) & = & V_X(1, 0, \ldots) \\
\mathrm{H}^4(X) & = & V_X(2, 0, \ldots) \oplus V_X(1, 0, \ldots) \oplus \mathbf{1}_X \\
\mathrm{H}^6(X) & = & V_X(3, 0, \ldots) \oplus V_X(1, 1, 0 \ldots) \oplus V_X(1, 0, \ldots) \oplus \mathbf{1}_X.
\end{array}$$

The trivial factor in $\mathrm{H}^4(X)$ corresponds to the Chern class $c_2(X)$; our main task is to analyze the trivial factor in $\mathrm{H}^6(X)$.

5. Cohomology computations for length-three subschemes

The general rule for multiplication in $A\{\mathfrak{S}_n\}$ is fairly complicated, so we will only give a formula in the case $(n = 3)$ we need. The fact that A only has terms of even degree simplifies the expressions of [13, 2.17]:

$$
\begin{aligned}
(\alpha_{\{1,2\}} \otimes \beta_{\{3\}})(12) \cdot (\gamma_{\{1,3\}} \otimes \delta_{\{2\}})(13) &= \alpha\beta\gamma\delta(132) \\
(\alpha_{\{1,2\}} \otimes \beta_{\{3\}})(12) \cdot (\gamma_{\{1,2\}} \otimes \delta_{\{3\}})(12) &= \Delta_*(\alpha\gamma) \otimes (\beta\delta)(\mathrm{id}) \\
\alpha_{\{1,2,3\}}(123) \cdot \beta_{\{1,2,3\}}(123) &= (\alpha\beta e)(132) \\
\alpha_{\{1,2,3\}}(123) \cdot \beta_{\{1,2,3\}}(132) &= (\Delta_*(\alpha\beta))_{\{1,2,3\}}(\mathrm{id}),
\end{aligned}
$$

where Δ_* is the adjoint of the threefold multiplication $A \otimes A \otimes A \to A$.

The remaining products can be deduced as formal consequences using the associativity of the multiplication, e.g.,

$$
\begin{aligned}
(\alpha_{\{1,2\}} &\otimes \beta_{\{3\}})(12) \cdot \gamma_{\{1,2,3\}}(132) \\
&= (\alpha_{\{1,2\}} \otimes \beta_{\{3\}})(12) \cdot (\gamma_{\{1,2\}} \otimes 1_{\{3\}})(12) \cdot (13) \\
&= (\Delta_*(\alpha\gamma)_{\{1,2\}} \otimes \beta_{\{3\}})(\mathrm{id}) \cdot (1_{\{1,3\}} \otimes 1_{\{2\}})(13) \\
&= \alpha\beta\gamma(\Delta_*(1))_{\{1,3\},\{2\}}(13),
\end{aligned}
$$

where α, β, and γ act on the diagonal via either the first or second variable. Thus in particular

$$
(12) \cdot (132) = (\Delta_*(1))_{\{1,3\},\{2\}}(13).
$$

We compute intersections among the absolute Hodge classes for $S^{[3]}$, i.e., classes that are Hodge for general K3 surfaces S. From now on, to condense notation we omit factors of the form $1_{\{i\}}, 1_{\{i,j\}}$, etc. from our expressions.

Based on the representation-theoretic analysis in Section 4, we expect one independent classes in codimension one, three in codimension two, and three in codimension three. We have the unique divisor

$$
\delta = (12) + (13) + (23).
$$

In codimension two, we have

$$
\begin{aligned}
P &= [\mathrm{pt}]_{\{1\}} + [\mathrm{pt}]_{\{2\}} + [\mathrm{pt}]_{\{3\}} \\
Q &= \sum_{j=1}^{22} e_{j\{1\}} \otimes e^{\vee}_{j\,\{2\}} + e_{j\{1\}} \otimes e^{\vee}_{j\,\{3\}} + e_{j\{2\}} \otimes e^{\vee}_{j\,\{3\}} \\
R &= (132) + (123).
\end{aligned}
$$

In codimension three, we have

$$
\begin{aligned}
U &= [\mathrm{pt}]_{\{1,2\}}(12) + [\mathrm{pt}]_{\{1,3\}}(13) + [\mathrm{pt}]_{\{2,3\}}(23) \\
V &= [\mathrm{pt}]_{\{3\}}(12) + [\mathrm{pt}]_{\{2\}}(13) + [\mathrm{pt}]_{\{1\}}(23) \\
W &= \sum_{j=1}^{22} e_{j\{1,2\}} \otimes e^{\vee}_{j\,\{3\}}(12) + e_{j\{1,3\}} \otimes e^{\vee}_{j\,\{2\}}(13) + e_{j\{2,3\}} \otimes e^{\vee}_{j\,\{1\}}(23).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\delta^2 &= (\Delta_*1)_{\{1,2\}}(12) + (\Delta_*1)_{\{1,3\}}(13) + (\Delta_*1)_{\{2,3\}}(23) \\
&\quad + 3((132) + (123)) \\
&= -2P - Q + 3R.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\delta \cdot P &= ((12) + (13) + (23)) \cdot ([\mathrm{pt}]_{\{1\}} + [\mathrm{pt}]_{\{2\}} + [\mathrm{pt}]_{\{3\}}) \\
&= 2U + V \\
\delta \cdot Q &= ((12) + (13) + (23)) \cdot (\sum_{j=1}^{22} e_{j\{1\}} \otimes e_{j\,\{2\}}^{\vee} + e_{j\{1\}} \otimes e_{j\,\{3\}}^{\vee} \\
&\quad + e_{j\{2\}} \otimes e_{j\,\{3\}}^{\vee}) \\
&= 22([\mathrm{pt}]_{\{1,2\}}(12) + [\mathrm{pt}]_{\{1,3\}}(13) + [\mathrm{pt}]_{\{2,3\}}(23)) \\
&\quad +2(\sum_{j=1}^{22} e_{j\{1,2\}} \otimes e_{j\,\{3\}}^{\vee} + e_{j\{1,3\}} \otimes e_{j\,\{2\}}^{\vee} + e_{j\{2,3\}} \otimes e_{j\,\{1\}}^{\vee}) \\
&= 22U + 2W \\
\delta \cdot R &= ((12) + (13) + (23))((132) + (123)) \\
&= 2(\Delta_*1_{\{1,2\},\{3\}}(12) + \Delta_*1_{\{1,3\},\{2\}} + \Delta_*1_{\{2,3\},\{1\}}) \\
&= -2(U + V + W).
\end{aligned}
$$

We deduce then that

$$
\delta^3 = \delta(-2P - Q + 3R) = -32U - 8V - 8W.
$$

Finally, we compute the intersection pairing on the subspace of the middle cohomology spanned by U, V, and W. Dimensional considerations give vanishing

$$
U^2 = V^2 = U \cdot W = V \cdot W = 0.
$$

For the remaining numbers, we get

$$
\begin{aligned}
U \cdot V &= ([\mathrm{pt}]_{\{1,2\}}(12) + [\mathrm{pt}]_{\{1,3\}}(13) + [\mathrm{pt}]_{\{2,3\}}(23)) \\
&\qquad \cdot ([\mathrm{pt}]_{\{3\}}(12) + [\mathrm{pt}]_{\{2\}}(13) + [\mathrm{pt}]_{\{1\}}(23)) \\
&= -3[\mathrm{pt}]_{\{1\}} \otimes [\mathrm{pt}]_{\{2\}} \otimes [\mathrm{pt}]_{\{3\}}\mathrm{id}
\end{aligned}
$$

and

$$
\begin{aligned}
W^2 &= (\sum_{j=1}^{22} e_{j\{1,2\}} \otimes e_{j\,\{3\}}^{\vee}(12) + e_{j\{1,3\}} \otimes e_{j\,\{2\}}^{\vee}(13) + e_{j\{2,3\}} \otimes e_{j\,\{1\}}^{\vee}(23))^2 \\
&= -3 \cdot 22 \cdot [\mathrm{pt}]_{\{1\}} \otimes [\mathrm{pt}]_{\{2\}} \otimes [\mathrm{pt}]_{\{3\}}\mathrm{id}.
\end{aligned}
$$

Remark 5.1. As a consistency check, we evaluate

$$
\begin{aligned}
\delta^6 &= (-32U - 8V - 8W)^2 = 2^6(8UV + W^2) \\
&= 2^6(-24 - 66)[\mathrm{pt}]_{\{1\}} \otimes [\mathrm{pt}]_{\{2\}} \otimes [\mathrm{pt}]_{\{3\}}\mathrm{id}.
\end{aligned}
$$

Using the formula for the point class (Equation 3), we obtain

$$
\delta^6 = -\frac{2^7 \cdot 3^2 \cdot 5}{2 \cdot 3} = -2^6 \cdot 3 \cdot 5.
$$

This is compatible with the Fujiki-type identity

$$D^6 = 15\,(D, D)^3, \quad D \in H^2(S^{[3]}, \mathbb{Q}),$$

as $(\delta, \delta) = -4$.

6. Evaluation of the distinguished absolute Hodge class

Let S be a general K3 surface and X a general deformation of $S^{[3]}$. The computations above show that the middle cohomology of X admits one Hodge class

$$H^6(X, \mathbb{Q}) \cap H^{3,3}(X) = \mathbb{Q}\eta$$

and the middle cohomology of $S^{[3]}$ admits three Hodge classes

$$H^6(S^{[3]}, \mathbb{Q}) \cap H^{3,3}(S^{[3]}) = \mathbb{Q}\eta \oplus \mathbb{Q}\delta^3 \oplus \mathbb{Q}c_2(X)\delta.$$

Our goal is to compute the self-intersection of η, at least up to the square of a rational number. Note that η is orthogonal to δ^3 and $\delta c_2(X)$ under the intersection form, by the analysis in Section 4. The analysis here gives the one structure constant left open in [15, Ex. 14].

Proposition 6.1. *Let X be deformation equivalent to $S^{[3]}$, for S a K3 surface. Let $\eta \in H^6(X, \mathbb{Q})$ denote the unique (up to scalar) absolute Hodge class. Then $\eta^2 = -3 \cdot 443$.*

Proof. The argument relies heavily on the analysis in Section 5. We extract the decomposable classes in codimension three. We have δ^3 already and

$$\delta \cdot P = 2U + V.$$

Hence the subspace span$\{2U + V, V - W\}$ is spanned by decomposable classes and has orthogonal complement spanned by $2U - V + 11W$. Thus we have

$$\eta = 2U - V + 11W$$

and

$$\begin{aligned} \eta^2 &= -4UV + 121W^2 \\ &= (12 - 121 \times 66)([\mathrm{pt}] \otimes [\mathrm{pt}] \otimes [\mathrm{pt}])\mathrm{id} \\ &= -3 \cdot 443. \end{aligned}$$

$\square$

7. Proof of the main theorem

We compute the cohomology class of a Lagrangian subspace $\mathbb{P}^3 \subset X$, where X is deformation equivalent to the Hilbert scheme of length three subschemes. As we shall see, the formula for $[\mathbb{P}^3]$ involves only decomposable classes, and not the absolute Hodge class η:

Lemma 7.1. *Let $\mathbb{P}^n \subset X$ be embedded in a general irreducible holomorphic symplectic variety of dimension $2n$. Then we have*

$$c_{2j}(\mathcal{T}_X|\mathbb{P}^n) = (-1)^j h^{2j} \binom{n+1}{j},$$

where h is the hyperplane class.

This is proved using the exact sequence

$$0 \to \mathcal{T}_{\mathbb{P}^n} \to \mathcal{T}_X|\mathbb{P}^n \to \mathcal{N}_{\mathbb{P}^n/X} \to 0$$

and

$$\mathcal{N}_{\mathbb{P}^n/X} \simeq \mathcal{T}^{\vee}_{\mathbb{P}^n},$$

reflecting the fact that $\mathbb{P}^n$ is a Lagrangian subvariety of X.

Regarding

$$\mathrm{H}^2(X,\mathbb{Z}) \subset \mathrm{H}_2(X,\mathbb{Z})$$

as a subgroup of index four, we can express $\ell = \lambda/4$ for some divisor class $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$. (This might not be primitive.)

Given a deformation of X such that λ remains algebraic, the subvariety $\mathbb{P}^3$ deforms as well [9]. Without loss of generality, we can deform X to a variety containing a $\mathbb{P}^3$, but otherwise having a general Hodge structure. In particular, we have a injection

$$\mathrm{Sym}(\mathrm{H}^2(X,\mathbb{Q})) \hookrightarrow \mathrm{H}^*(X,\mathbb{Q}).$$

We expect to be able to write

$$\left[\mathbb{P}^3\right] = a\lambda c_2(X) + b\lambda^3 + d\eta$$

for some $a, b, d \in \mathbb{Q}$.

Furthermore, the Fujiki relations [5] imply that for each $f \in \mathrm{H}^2(X,\mathbb{Z})$,

$$f^6 = e_0 \left(f, f\right)^3, \quad c_2(X)f^4 = e_2 \left(f, f\right)^2, \quad c_4(X)f^2 = e_4 \left(f, f\right)$$

for suitable rational constants e_0, e_2, e_4. Precisely, we have [4]

$$c_2^2(X)f^2 = \frac{5}{2}c_4(X)f^2.$$

The Riemann-Roch formula gives

$$\chi(\mathcal{O}_X(f)) = \frac{f^6}{6!} + \frac{c_2(X)f^4}{12 \cdot 4!} + \frac{f^2(3c_2^2 - c_4)}{720 \cdot 2!} + 4.$$

On the other hand, we know that

$$\chi(\mathcal{O}_X(f)) = \frac{1}{3!2^3}((f,f)+8)((f,f)+6)((f,f)+4).$$

Perhaps the quickest way to check this formula is to observe that if $X = S^{[3]}$ and f is a very ample divisor on S with no higher cohomology then the induced sheaf $\mathcal{O}_X(f)$ has no higher cohomology and

$$\dim \Gamma(\mathcal{O}_X(f)) = \dim \mathrm{Sym}^3(\Gamma(\mathcal{O}_S(f))) = \binom{\chi(\mathcal{O}_S(f)) + 2}{3}.$$

Equating coefficients, we find

$$\begin{array}{rcl} f^6 & = & 15\,(f,f)^3 \\ f^4 c_2 & = & 108\,(f,f)^2 \\ f^2 c_4 & = & 480\,(f,f) \\ f^2 c_2^2 & = & 1200\,(f,f) \end{array}$$

We generate Diophantine equations for $a, b, (\lambda, \lambda)$ and eventually, d. First, observe that

$$(\lambda, \ell) = \lambda \cdot \ell = \deg \lambda|\mathbb{P}^3$$

so that $\lambda|\mathbb{P}^3$ is $(\lambda, \lambda)/4$ times the hyperplane class. Thus we have

$$\left[\mathbb{P}^3\right] \lambda^3 = ((\lambda, \lambda)/4)^3$$

and

$$\left[\mathbb{P}^3\right] \lambda^3 = a\lambda^4 c_2(X) + b\lambda^6.$$

Equating these expressions and evaluating the terms, we find

$$(\lambda, \lambda)\,(15b - 1/64) + 108a = 0.$$

We have divided out by (λ, λ); the solution $(\lambda, \lambda) = 0$ is not possible for geometric reasons, and we shall exclude it algebraically below.

Second, the Lemma on restrictions of Chern classes implies

$$\left[\mathbb{P}^3\right] \lambda c_2(X) = -\,(\lambda, \lambda)$$

whereas the formula for the class of $\mathbb{P}^3$ yields

$$\left[\mathbb{P}^3\right] \lambda c_2(X) = a\lambda^2 c_2(X)^2 + b\lambda^4 c_2(X).$$

Thus we obtain

$$108b\,(\lambda, \lambda) + (1200a + 1) = 0.$$

Remark 7.2. The cup product of $\mathrm{H}^*(X)$ is compatible with the G_X-action, so the subring generated by Chern classes and elements of $\mathrm{H}^2(X)$ is orthogonal to η. Thus even if the decomposition of $[\mathbb{P}^3]$ were to involve η, the computations up to this point would not reflect this.

Finally, the fact that

$$\left[\mathbb{P}^3\right]^2 = c_3(\mathcal{N}_{\mathbb{P}^3/X}) = c_3(\mathcal{T}_{\mathbb{P}^3}^\vee) = -4$$

yields the *cubic* equation

$$15b^2\,(\lambda, \lambda)^3 + 216ab\,(\lambda, \lambda)^2 + 1200\,(\lambda, \lambda)\,a^2 + d^2 \eta \cdot \eta = -4.$$

Proposition 6.1 implies that $\eta \cdot \eta = -11 \cdot 443$. In particular, $(\lambda, \lambda) = 0$ is excluded.

Eliminating a and b from these equations and setting $L = (\lambda, \lambda)$, we obtain

$$(4) \qquad 2^{14} \cdot 3^2 \cdot 11 \cdot 443 d^2 = 5^2 L^3 + 2^5 \cdot 3^2 L^2 + 2^8 \cdot 5L + 2^{16} \cdot 3 \cdot 11.$$

We know, *a priori*, that $L \in \mathbb{Z}$ and $d \in \mathbb{Q}$.

Proposition 7.3. *The only solution to (4) with $L \in \mathbb{Z}$ and $d \in \mathbb{Q}$ is $d = 0$ and $L = -48$.*

We assume this for the moment; its proof can be found in Section 8. Back-substitution yields

$$a = 1/96, \quad b = 1/384, \quad (\ell, \ell) = -3.$$

We claim that $\lambda/2 \in \mathrm{H}^2(X, \mathbb{Z})$, i.e., λ is not primitive. Using the isomorphism

$$\mathrm{H}_2(X, \mathbb{Z}) = \mathrm{H}_2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta^\vee, \quad (\delta^\vee, \delta^\vee) = -1/4$$

we can express

$$\ell = D + m\delta^\vee, \quad D \in \mathrm{H}_2(S, \mathbb{Z}), m \in \mathbb{Z}.$$

If λ were primitive then m would have to be odd and

$$-3 = (\ell, \ell) = (D, D) - m^2/4.$$

Since $(D, D) \in 2\mathbb{Z}$, we have a contradiction.

8. Diophantine analysis

Theorem 8.1. *The only solution to*

$$2^{14} \cdot 3^2 \cdot 11 \cdot 443 d^2 = 5^2 L^3 + 2^5 \cdot 3^2 L^2 + 2^8 \cdot 5L + 2^{16} \cdot 3 \cdot 11$$

with $L \in \mathbb{Z}$ and $d \in \mathbb{Q}$ is $L = -48$, $d = 0$.

Proof. Put $x = 2^{-4} \cdot 5^2 \cdot 11 \cdot 443(L + 48)$ and $y = 2 \cdot 3 \cdot 5^2 \cdot 11^2 \cdot 443^2 d$. The equation then takes the form

$$(5) \qquad\qquad E : y^2 = x^3 + ax^2 + bx$$

where

$$a = -3^2 \cdot 11 \cdot 23 \cdot 443, \qquad b = 2^2 \cdot 5^2 \cdot 11^3 \cdot 13 \cdot 443^2.$$

It suffices to prove the stronger statement that there are no solutions to (5) with $x, y \in \mathbb{Z}[\frac{1}{2}]$, apart from $x = y = 0$.

The proof is given in two steps. Proposition 8.2 below determines explicitly the structure of the Mordell–Weil group $E(\mathbb{Q})$. Proposition 8.3 then identifies the integral points. $\square$

Algorithms for both of these steps are implemented in computer algebra systems such as `Sage` [20] and `Magma` [2], and the theorem may be verified this way. To avoid depending on the correctness of these systems, we give alternative proofs that use as little machine assistance as possible. The only step that is perhaps unreasonable to verify by hand is that a certain point P with large coordinates (about 30 digits) lies in $E(\mathbb{Q})$.

We first set notation and briefly recall some facts about point multiplication on elliptic curves. Let O denote the zero element of $E(\mathbb{Q})$ (the point at infinity). For nonzero $R \in E(\mathbb{Q})$ we write

$$R = (x(R), y(R)) = \left(\frac{\alpha(R)}{e(R)^2}, \frac{\beta(R)}{e(R)^3} \right),$$

where $\alpha, \beta, e \in \mathbb{Z}$, $e \geq 1$ and $(\alpha, e) = (\beta, e) = 1$.

Let $R \in E(\mathbb{Q})$, $R \neq O$. If p is a prime, then $p \mid e(R)$ if and only if R reduces to the identity in $E(\mathbb{F}_p)$. If $m \geq 1$ and $mR \neq O$, then $e(R) \mid e(mR)$. For $m = 2$ we have the following formula:
(6)
$$x(2R) = \frac{\alpha(2R)}{e(2R)^2} = \frac{(\alpha(R)^2 - b \cdot e(R)^4)^2}{4e(R)^2 (\alpha(R)^3 + a \cdot \alpha(R)^2 e(R)^2 + b \cdot \alpha(R) e(R)^4)}.$$

Moreover, if R reduces to a nonsingular point in $E(\mathbb{F}_p)$, then p cannot divide both the numerator and denominator of the fraction on the right side of (6). In other words, there is no cancellation locally at p. One proof of this is given in [23, Prop. IV.2]; as pointed out in that paper, it can also be proved from properties of real-valued non-archimedean local heights.

The discriminant of the Weierstrass equation (5) is given by

$$\Delta = 16b^2(a^2 - 4b) = -2^8 \cdot 5^4 \cdot 11^8 \cdot 13^2 \cdot 113 \cdot 127 \cdot 443^6,$$

so the model is minimal, and the primes of bad reduction are 2, 5, 11, 13, 113, 127 and 443. For $p = 2, 5, 11, 13, 443$, we have that $p \mid \alpha(R)$ if and only if R reduces to a singular point of $E(\mathbb{F}_p)$, i.e. the only singular point of $E(\mathbb{F}_p)$ is $(0 : 0 : 1)$ for these primes. The point $Q = (0,0)$ has order two, and addition with Q is given by the formula

(7)
$$R + Q = \left(\frac{b}{x(R)}, \frac{-b \cdot y(R)}{x(R)^2} \right).$$

Proposition 8.2. *We have $E(\mathbb{Q}) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$, where the free part is generated by the point P with coordinates*

$$\left(\frac{2 \cdot 3^2 \cdot 11^2 \cdot 83^2 \cdot 443^2 \cdot 6481^2}{7^4 \cdot 41^2 \cdot 71^2 \cdot 193^2}, \frac{2 \cdot 3 \cdot 11^3 \cdot 31 \cdot 83 \cdot 163 \cdot 443^2 \cdot 6481 \cdot 240623 \cdot 3691717}{7^6 \cdot 41^3 \cdot 71^3 \cdot 193^3} \right)$$

and the torsion part by $Q = (0, 0)$.

Proof. We first check that the torsion subgroup is as described. We have $E(\mathbb{F}_3) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $E(\mathbb{F}_{19}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$. For ℓ prime, by [18, Prop. VII.3.1] we see that $E(\mathbb{Q})[\ell]$ injects into $E(\mathbb{F}_3)$ for $\ell \neq 3$ and that $E(\mathbb{Q})[\ell]$ injects into $E(\mathbb{F}_{19})$ for $\ell \neq 19$. These facts force $E(\mathbb{Q})[2] = \mathbb{Z}/2\mathbb{Z}$, $E(\mathbb{Q})[3] = 0$, and $E(\mathbb{Q})[\ell] = 0$ for $\ell \neq 2, 3$. Hence $E_{\text{tors}}(\mathbb{Q}) = \langle Q \rangle$.

Now we consider the free part. The point P was found using Cremona's mwrank library [3] included in `Sage` [20]. We may check that $P \in E(\mathbb{Q})$ using a computer; this shows that $\text{rank}\, E \geq 1$. (The point P is reasonably difficult to find from scratch; indeed the standard functions for computing $E(\mathbb{Q})$ in both `Magma` and `Sage` fail to find P.)

To show that $\text{rank}\, E \leq 1$ we use a standard 2-descent strategy (see for example [19, Ch. III]). Consider the auxiliary curve

$$E' : y^2 = x^3 - 2ax^2 + (a^2 - 4b)x.$$

There are isogenies $\phi : E \to E'$ and $\hat{\phi} : E' \to E$ of degree 2, and injections

$$E(\mathbb{Q})/\hat{\phi}(E'(\mathbb{Q})) \xrightarrow{\psi} S \subset \mathbb{Q}^*/(\mathbb{Q}^*)^2,$$

$$E'(\mathbb{Q})/\phi(E(\mathbb{Q})) \xrightarrow{\psi'} S' \subset \mathbb{Q}^*/(\mathbb{Q}^*)^2,$$

where S consists of the cosets $\delta(\mathbb{Q}^*)^2$ for $\delta \mid 2 \cdot 5 \cdot 11 \cdot 13 \cdot 443$, and S' of the cosets for $\delta \mid 11 \cdot 113 \cdot 127 \cdot 443$ (these are the primes dividing b and $a^2 - 4b$ respectively). We must determine which elements of S and S' arise from points in $E(\mathbb{Q})$ and $E'(\mathbb{Q})$. This is achieved by testing for the existence of rational points on the two families of quartic curves

$$C_\delta : \delta w^2 = \delta^2 z^4 + \delta a z^2 + b, \qquad \delta \in S,$$

(8) $$C'_\delta : \delta w^2 = \delta^2 z^4 - 2\delta a z^2 + (a^2 - 4b), \qquad \delta \in S'.$$

We first consider the C'_δ. If $443 \mid \delta$ then (8) has no solution in $\mathbb{Q}_{443}$; if $(\delta/5) = -1$ then it has no solution in $\mathbb{Q}_5$; and if $\delta \neq 1 \pmod 8$ then it has no solution in $\mathbb{Q}_2$. These conditions rule out all but $\delta = 1$ and $\delta = -113 \cdot 127$. These correspond to the classes in $E'(\mathbb{Q})/\phi(E(\mathbb{Q}))$ of O and the unique two-torsion point of $E'(\mathbb{Q})$; both have trivial image in $\hat{\phi}(E'(\mathbb{Q}))/2E(\mathbb{Q})$.

Now we examine the C_δ. For $\delta = 11 \cdot 13$ there is the trivial rational point $z = 0$, $w = 2 \cdot 5 \cdot 11 \cdot 443$, corresponding to the class of Q in $E(\mathbb{Q})/\hat{\phi}(E'(\mathbb{Q}))$. For $\delta = 2$ there is a (highly nontrivial) rational point

corresponding to P, namely $z = (\frac{1}{2}x(P))^{1/2}$, $w = y(P)(2x(P))^{1/2}$. Rational points are automatic for $\delta = 1$ and $\delta = 2 \cdot 11 \cdot 13$ since the image of ψ is a subgroup of S. We will show that $C_\delta(\mathbb{Q}) = \emptyset$ for all other δ.

Rewriting the equation for C_δ as $4\delta w^2 = (2\delta z^2 + a)^2 - (a^2 - 4b)$, we see immediately that $\delta > 0$ since $a^2 - 4b < 0$. Next, note that $(p/113) = 1$ for $p = 2, 11, 13, 443$, but $(5/113) = -1$. Thus if $5 \mid \delta$ we have $(\delta/113) = -1$; this is impossible as $v_{113}(a^2 - 4b) = 1$. Therefore $5 \nmid \delta$.

To finish the argument for the C_δ it suffices to show that $C_\delta(\mathbb{Q}) = \emptyset$ for $\delta = 11$, 443 and $11 \cdot 443$; the statement for the remaining δ will then follow automatically from the subgroup property.

Let $\delta = 11$, 443, or $11 \cdot 443$. Let $u = z^2$ and let $(u, w) = (u_0/t, w_0/t)$ be a rational point on the conic $\delta w^2 = \delta^2 u^2 + \delta a u + b$, where $u_0, w_0, t \in \mathbb{Z}$. Intersecting the conic with a line of slope X/Y through $(u_0/t, w_0/t)$, we obtain the parameterization $z^2 = f(X, Y)/g(X, Y)$ where

$$f(X, Y) = u_0 X^2 - 2w_0 XY + (ta + \delta u_0)Y^2,$$
$$g(X, Y) = t(X^2 - \delta Y^2),$$

and where we may assume that $X, Y \in \mathbb{Z}$ and $(X, Y) = 1$. Taking resultants, we find that any prime p dividing $f(X, Y)$ and $g(X, Y)$ must divide t or $a^2 - 4b = -11^2 \cdot 113 \cdot 127 \cdot 443^2$. Thus

$$(9) \qquad\qquad\qquad f(X, Y) = \varepsilon Z^2,$$
$$(10) \qquad\qquad\qquad g(X, Y) = \varepsilon W^2$$

for some $\varepsilon \mid 11 \cdot 113 \cdot 127 \cdot 443t$, and some $W, Z \in \mathbb{Z}$. We now consider each δ in turn, summarizing the local obstructions encountered for each possible ε.

Let $\delta = 11$. We take $u_0 = 3 \cdot 5^2 \cdot 443$, $w_0 = 2^2 \cdot 5 \cdot 11 \cdot 443$, $t = 1$. Then $\varepsilon \mid 11 \cdot 113 \cdot 127 \cdot 443$. If $443 \mid \varepsilon$ then (10) has no solution in $\mathbb{Q}_{443}$. If $11 \mid \varepsilon$ then (9) has no solution in $\mathbb{Q}_{11}$. If $(\varepsilon/11) = -1$ then (10) has no solution in $\mathbb{Q}_{11}$. This leaves $\varepsilon \in \{1, 113, -127, -113 \cdot 127\}$. For these ε we have $(\varepsilon/443) = 1$. Equation (9) implies that $X = 14Y$ or $X = 110Y$ (mod 443); both options contradict (10).

Now consider $\delta = 443$. We take $u_0 = -3 \cdot 11 \cdot 13$, $w_0 = 11 \cdot 13 \cdot 443$, $t = 2$. Then $\varepsilon \mid 2 \cdot 11 \cdot 113 \cdot 127 \cdot 443$. Suppose that $443 \nmid \varepsilon$. If $(\varepsilon/443) = 1$ then (10) has no solution in $\mathbb{Q}_{443}$, and if $(\varepsilon/443) = -1$ then (9) has no solution in $\mathbb{Q}_{443}$. Now let $\varepsilon = 443\varepsilon'$. If $(\varepsilon'/443) = -1$ then (10) has no solution in $\mathbb{Q}_{443}$. Now assume that $(\varepsilon'/443) = 1$. Observe that $(p/443) = (p/11)$ for $p \in \{-1, 2, 113, 127\}$, but $(11/443) = -1$. This implies that either $11 \nmid \varepsilon'$ and $(\varepsilon'/11) = 1$, or $11 \mid \varepsilon'$ and $(\frac{\varepsilon'}{11}/11) = -1$. In both cases, (9) forces $Y = 10X$ (mod 11), and this contradicts (10).

Finally let $\delta = 11 \cdot 443$. We take $u_0 = 5^3 \cdot 11$, $w_0 = 2 \cdot 5 \cdot 11 \cdot 443$, $t = 3^2$. Then $\varepsilon \mid 3 \cdot 11 \cdot 113 \cdot 127 \cdot 443$. If $11 \nmid \varepsilon$ then there are no solutions to (9) in $\mathbb{Q}_{11}$. Suppose that $11 \mid \varepsilon$. Then since $(p/13) = 1$ for $p \in \{-1, 3, 113, 127, 443\}$ and $(11/13) = -1$, we have $(\varepsilon/13) = -1$; then (9) has no solution in $\mathbb{Q}_{13}$.

This completes the 2-descent. In particular, we have found that

$$|E(\mathbb{Q})/2E(\mathbb{Q})| = |E(\mathbb{Q})/\hat{\phi}(E'(\mathbb{Q}))| \cdot |\hat{\phi}(E'(\mathbb{Q}))/2E(\mathbb{Q})| = 4 \cdot 1 = 4,$$

and that $E(\mathbb{Q})/2E(\mathbb{Q})$ is generated by P and Q. Moreover, for $R \neq O, Q$ the image of $x(R)$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ is one of $\{1, 2, 11 \cdot 13, 2 \cdot 11 \cdot 13\}$.

At this stage we know that $\langle P, Q \rangle$ is of finite index in $E(\mathbb{Q})$; we must still check that it exhausts $E(\mathbb{Q})$. Suppose not; then for some prime ℓ and some $R \in E(\mathbb{Q})$ we have $\ell R = P$ or $\ell R = P + Q$. We cannot have $2R = P$ as P is not divisible by 2 in $E(\mathbb{F}_3)$; similarly $2R = P + Q$ is excluded by considering $E(\mathbb{F}_7)$. Thus we may assume that ℓ is odd. If $\ell R = P + Q$ we replace R by $R + Q$, so now may assume that $\ell R = P$ and $\ell(R + Q) = P + Q$.

In this case $e(R) \mid e(P) = 7^2 \cdot 41 \cdot 71 \cdot 193$. From (7) we have

$$x(P + Q) = \frac{2 \cdot 5^2 \cdot 7^4 \cdot 11 \cdot 13 \cdot 41^2 \cdot 71^2 \cdot 193^2}{3^2 \cdot 83^2 \cdot 6481^2},$$

so similarly $e(R + Q) \mid 3 \cdot 83 \cdot 6481$. Moreover by (7) we have

$$\alpha(R)\alpha(R + Q) = b \cdot e(R)^2 e(R + Q)^2.$$

Since $(\alpha(R), e(R)) = (\alpha(R+Q), e(R+Q)) = 1$ this implies that $\alpha(R) = b_1 e(R + Q)^2$ and $\alpha(R + Q) = (b/b_1)e(R)^2$ for some $b_1 \mid b$. Since P has singular reduction at $p = 2, 11, 443$, so does R, so $2 \cdot 11 \cdot 443 | b_1$. Similarly we find that $2 \cdot 5 \cdot 11 \cdot 13 \mid (b/b_1)$. Comparing with the classes of $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ found by the 2-descent shows that we must have $b_1 = 2 \cdot 11^2 \cdot 443^2$ and $b/b_1 = 2 \cdot 5^2 \cdot 11 \cdot 13$.

At this point we have reduced to 24 possibilities for $e(R)$ and 8 possibilities for $\alpha(R)$, and it is straightforward to check using a computer that the only pair defining a point on $E(\mathbb{Q})$ is $R = P$. Alternatively one may finish the argument using congruences. We sketch one quick way to do it: first prove that $3 \mid \alpha(R)$ by considering images in $E(\mathbb{F}_3)$. Then for only 10 remaining values of $x(R)$ is $x^3 + ax^2 + bx$ a square in $\mathbb{Q}_{11}$, and for only one of these is it a square in $\mathbb{Q}_{31}$. $\square$

Proposition 8.3. *The only solution to (5) with $x, y \in \mathbb{Z}[\frac{1}{2}]$ is $x = y = 0$.*

Proof. Let $n \in \mathbb{Z}$, $k \in \{0, 1\}$. We must prove that $x(nP + kQ) \notin \mathbb{Z}[\frac{1}{2}]$ for $n \neq 0$. We consider several cases.

First suppose that $k = 0$ and $n \neq 0$. Since $7 \mid e(P)$, also $7 \mid e(nP)$, so $x(nP) \notin \mathbb{Z}[\frac{1}{2}]$.

Next suppose that $k = 1$ and that n is odd. Since $3 \mid e(P + Q)$, we have $3 \mid e(n(P + Q)) = e(nP + Q)$, so $x(nP + Q) \notin \mathbb{Z}[\frac{1}{2}]$.

Now suppose that $k = 1$ and $n = 2r$ where r is odd. Since $79 \mid e(2P + Q)$, we have $79 \mid e(r(2P + Q)) = e(nP + Q)$, so $x(nP + Q) \notin \mathbb{Z}[\frac{1}{2}]$.

The last case is $k = 1$, $n = 0 \pmod 4$, $n \neq 0$. Write $n = 2^i r$ for some $i \geq 2$ and odd r. To continue the pattern we must find a prime q playing the same role as 79 from the previous case. For this, we first establish that

$$(11) \qquad \alpha(2^j P) = \pm 4 \pmod 7 \qquad \text{for } j \geq 2.$$

Indeed, one checks that $4P$ has nonsingular reduction for all p. The doubling formula (6) and the comments regarding cancellation immediately following it then imply that $\alpha(2^{j+1}P) = \pm(\alpha(2^j P)^2 - be(2^j P)^4)^2$ for all $j \geq 2$. Since $7 \mid e(2^j P)$ and $\alpha(4P) = \pm 4 \pmod 7$, identity (11) follows by induction.

In particular $\alpha(2^i P) = \pm 4 \pmod 7$, so there must exist some prime q, not congruent to 1 modulo 7, dividing $\alpha(2^i P)$. We cannot have $q = 113$ or $q = 127$, as both of these are 1 (mod 7). Also, $q \notin \{2, 5, 11, 13, 443\}$, since for all of these primes the point $(0 : 0 : 1)$ is singular in $E(\mathbb{F}_p)$, whereas $2^i P$ has nonsingular reduction for all p. Therefore q is not a prime of bad reduction. From (7) we obtain $q \mid e(2^i P + Q)$. Finally, since $nP + Q = r(2^i P + Q)$, we have also $q \mid e(nP + Q)$, so that $x(nP + Q) \notin \mathbb{Z}[\frac{1}{2}]$. $\qquad \square$

Remark 8.4. In several places in the above proof we use certain facts about $2P$ and $4P$. It is not necessary to compute their full coordinates, which are quite large (for example $\alpha(4P)$ has 256 digits). In every case it is possible to work p-adically to low precision. For example, to check that $4P$ has nonsingular reduction at 2, it suffices to apply the doubling formula twice, using as input $a = 1 \pmod{2^3}$, $b = 28 \pmod{2^5}$ and $x(P) = 2 \pmod{2^4}$, to find that $x(2P) = 4 \pmod{2^5}$ and $x(4P) = 2^{-4} \pmod{2^{-3}}$.

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